On Total 9-Coloring Planar Graphs of Maximum Degree Seven

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Abstract: Given a graph G, a total k-coloring of G is a simultaneous coloring of the vertices and edges of G with at most k colors. If $\Delta(G)$ is the maximum degree of G, then no graph has a total Δ-coloring, but Vizing conjectured that every graph has a total $(\Delta + 2)$-coloring. This Total Coloring Conjecture remains open even for planar graphs. This article proves one of the two remaining planar cases, showing that every planar (and projective) graph with $\Delta \leq 7$ has a total 9-coloring by means of the discharging method. © 1999 John Wiley & Sons, Inc. J Graph Theory 31: 67–73, 1999

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1. INTRODUCTION

Given a graph G, then an *element* of G is a member of $V(G) \cup E(G)$. Let two elements of a graph G be *adjacent* if they are either adjacent or incident in the

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traditional sense. Given a graph G, a *total k-coloring* of G is a function that takes each element to $\{1, \ldots, k\}$, such that adjacent distinct elements receive distinct colors. Given a graph G, let $\Delta(G)$ be the maximum vertex degree of G. If the graph is clear from the context, Δ will be used. Clearly, no graph has a total Δ-coloring. In 1964, Vizing [9] (see also [2]) made the following conjecture, known as the Total Coloring Conjecture.

Conjecture 1.1. *Every graph has a total $(\Delta + 2)$-coloring.*

This conjecture is trivial for $\Delta \leq 2$. Rosenfeld [7] and Vijayaditya [8] solved it for $\Delta = 3$. Kostochka solved the $\Delta = 4$ [5] and $\Delta = 5$ [6] cases.

The conjecture remains open even for planar graphs, but more is known. Borodin proved it for planar graphs with $\Delta \geq 9$. The $\Delta = 8$ case was solved for planar graphs by Yap [10] and Andersen [1]. Thus, the only two cases for planar graphs that remained were for $\Delta = 6$ and $\Delta = 7$.

This article proves that every planar graph with $\Delta \leq 7$ has a total 9-coloring. Thus, the only case remaining for planar graphs now is the $\Delta = 6$ case.

Section 2 describes thirteen special types of vertices. Section 3 uses the discharging method to prove that every planar graph (and every projective planar graph) has one of these special vertices. Section 4 shows that no graph that is minimal with respect to not having a total 9-coloring has one of these special vertices. This combines to give the main result, stated below.

Theorem 1.1. *Every planar or projective planar graph with $\Delta \leq 7$ has a total 9-coloring.*

The authors have hope that the discharging method will be able to complete the proof of the total coloring conjecture at least for planar and projective planar graphs.

2. SPECIAL VERTICES

To describe the structures of this section, some notation will be introduced. Let a *k-vertex* be a vertex of degree k. Let an *at most k-vertex*, or simply an *$(\leq k)$-vertex*, be a vertex of degree at most k. Let $(\geq k)$-vertex be defined analogously. Also, let k-face, $(\leq k)$-face, and $(\geq k)$-face be defined analogously.

Given a graph G and integers $i, j_1, \ldots, j_k$ with $2 \leq k \leq i$, let a $(j_1, \ldots, j_k)$ *around an* i be an i-vertex x of G such that, for each $m \in \{1, \ldots, k\}$, there is an $(\leq j_m)$-vertex y_m of G, the vertices $y_1, \ldots, y_k$ are distinct neighbors of x, and further, that for each $m \in \{1, \ldots, k-1\}, y_m$ is adjacent to y_{m+1}. Similarly, given a graph G and integers $i, j_1, \ldots, j_i$, let a $(j_1, \ldots, j_i)$ *surrounding an* i be an i-vertex x of G such that, for each $m \in \{1, \ldots, i\}$, there is an $(\leq j_m)$-vertex y_m of G, the vertices $y_1, \ldots, y_i$ are the neighbors of x, and further, that, for each $m \in \{1, \ldots, i\}, y_m$ is adjacent to $y_{(m \bmod i)+1}$. For instance, a (7, 7) around a 3 is a 3-vertex in a triangle with two (≤ 7)-vertices.

Let a vertex be *special* if it is one of the following (for now, ignore the second column):

an (≤ 2)-vertex x adjacent to an (≤ 7)-vertex y	$-xy$
a 3-vertex x adjacent to an (≤ 6)-vertex y	$-xy$
a 4-vertex x adjacent to an (≤ 5)-vertex y	$-xy$
a $(7,7)$ around a 3	$-xy_1$
a $(6,7)$ around a 4	$-xy_1$
a $(7,7,7)$ around a 4	$-xy_2$
a $(5,6,7)$ around a 5	$-xy_1$
a $(5,7,5)$ around a 5	$-xy_1-xy_3$
a $(5,7,7,7,5)$ around a 5	$+y_1y_5-xy_1-xy_5$
a $(5,7,6,7,6)$ around a 5	$+y_1y_5-xy_1-xy_3$
a $(5,7,7,6,6)$ around a 5	$+y_1y_5-xy_1-xy_4$
a $(6,6,7,7,7)$ surrounding a 5	$-xy_1-xy_2$
a $(7,6,7,6,7)$ surrounding a 5	$-xy_2-xy_4$

The next section will show that every planar or projective planar graph with $\Delta \leq 7$ has a special vertex.

3. DISCHARGING

This section will deal with connected graphs embedded on the plane and the projective plane. Given a graph G embedded on either of these surfaces, let v, e, and f be, respectively, the number of vertices, edges, and faces of this embedding of G. Euler's formula states that, if a connected graph G is embedded in the plane, then $v-e+f=2$, and if G is embedded in the projective plane (without loss of generality, each face is an open 2-cell, or this embedding can be interpreted to be on the plane), then $v-e+f=1$.

Let an embedded graph G be *charged*, if a function ch is defined as follows: for each member x of $V(G)\cup F(G)$, let $ch(x) := 4-\deg(x)$. The following lemma is the basic premise of the discharging method.

Lemma 3.1. *Let G be a charged (connected) graph, which is embedded in the plane or the projective plane. Then*

$$\sum_{x\in V(G)\cup F(G)} ch(x) \geq 4.$$

Proof. Assume that the lemma is false. Let G be a charged graph embedded on the plane or the projective plane not satisfying the conclusion of the lemma. Thus, Euler's formula states that $v-e+f\geq 1$. This may be restated as $4v-4e+4f \geq 4$, or $4v-2e+4f-2e\geq 4$. Since $\sum_{x\in V}\deg(x)=2e$, and $\sum_{A\in F}\deg(A)=2e$, it follows that $\sum_{x\in V}(4-\deg(x))+\sum_{A\in F}(4-\deg(A))\geq 4$, which is equivalent to the conclusion of the lemma. ∎

A corollary to Lemma 3.1 is the classic result that every graph embedded in the plane has either an (≤ 3)-vertex or a 3-face. This is true, because a positive sum must have a positive term. If a vertex x has $ch(x) > 0$, it is an (≤ 3)-vertex. If a face A has $ch(A) > 0$ (since the graph is simple), it is a 3-face. By *discharging*, or locally redistributing the positive charge away from (≤ 3)-vertices and 3-faces, one can prove the existence in a planar graph of other small structures. This is what is done next, to show that every planar (and projective planar) graph with $\Delta \leq 7$ has a special vertex.

Let a (j,k)-*edge* be an edge adjacent to a j-vertex and a k-vertex. Let an $(\leq j, \geq k)$-*edge* be defined in the obvious way.

Let a charged graph G be *discharged*, if a function ch' is defined by modifying ch according to Rules 1 to 5 below. It is convenient for counting purposes within the proof to describe some of the rules in the following manner: send k from x through y to z. The net effect of this rule decreases the charge at x by k, leaves the charge of y unchanged, and increases the charge of z by k. But for counting purposes, x sends the charge to y, and z receives the charge from y.

(1) For each (3, 7)-edge α, and for each (≥ 4)-face A adjacent to α, send $\frac{3}{14}$ from the 3-vertex adjacent to α through A to the 7-vertex adjacent to α.
(2) For each $(\leq 5, \geq 6)$-edge α adjacent to a 3-face A and an (≥ 4)-face B, send $\frac{1}{6}$ from A through B to the (≥ 6)-vertex adjacent to α.
(3) For each 3-face A, and for each 7-vertex x adjacent to A, send $\frac{3}{7}$ from A to x.
(4) For each 3-face A, and for each 6-vertex x adjacent to A, send $\frac{1}{3}$ from A to x.
(5) For each 3-face A adjacent to $k > 0$ 5-vertices, which, after applying Rules 1 to 4, still has charge $c > 0$, send $\frac{c}{k}$ from A to each 5-vertex adjacent to A.

Theorem 3.1. *Every graph G that is planar or projective planar and has* $\Delta(G) \leq 7$ *contains a special vertex.*

Proof. For a contradiction, assume that the theorem is false. Let G be a planar or projective planar graph with $\Delta \leq 7$. Embed G in the plane or the projective plane. Let G be charged, and then discharged.

By examining each face A according to its degree, it will be shown that $ch'(A) \leq 0$.

Let A_3 be a 3-face of G. Note that $ch(A_3) = 1$. If A_3 is adjacent to a 3-vertex, then that vertex is a (7, 7) around a 3 and is special.

If A_3 is adjacent to a 4-vertex x, then either x is a (6, 7) around a 4 and is special, or A_3 is adjacent to two 7-vertices and sends $\frac{3}{7}$ to each of them by Rule 3. Also, either x is a (7, 7, 7) around a 4 and is special, or A_3 sends $\frac{1}{3}$ to its adjacent faces by Rule 2, and $ch(A_3) = -\frac{4}{21}$.

If A_3 is adjacent to a 5-vertex, then it clearly sends out at least 1 by Rules 2 to 5, and $ch(A_3) \leq 0$. Thus, A_3 is adjacent to three (≥ 6)-vertices, and sends out at least 1 by Rules 3 and 4, and $ch(A_3) \leq 0$.

Let A be an (≥ 4)-face of G. Note that $ch(A) \leq 0$, and since no charge is sent into A (but possibly through it), $ch'(A) \leq 0$ as well.

Next, the vertices will be considered, with corresponding results.

Each (≤ 2)-vertex is special.

Let x_3 be a 3-vertex of G. Note that $ch(x_3) = 1$. If x_3 has a neighbor of degree at most 6, then x_3 is special. Thus, x_3 is adjacent to three 7-vertices. If x_3 is adjacent to a 3-face, then it is a (7, 7) around a 3 and is special. Thus, x_3 is adjacent to three (≥ 4)-faces. It follows that x_3 sends $\frac{3}{7}$ to each of those faces by Rule 1. Thus, $ch'(x_3) = -\frac{2}{7}$.

Let x_4 be a 4-vertex. Note that $ch(x_4) = 0$. Since the Rules do not affect this charge, $ch'(x_4) = 0$ as well.

Let x_5 be a 5-vertex. Note that $ch(x_5) = -1$. Also, the only charge that x_5 receives is by Rule 5.

Let A be a 3-face adjacent to x_5. Note that A is not adjacent to an (≤ 4)-vertex, or such a vertex would be special. If A is adjacent to no 7-vertex, then A sends at most $\frac{1}{3}$ into x_5. If A is adjacent to one 7-vertex and two 5-vertices, then A sends at most $\frac{2}{7}$ into x_5. If A is adjacent to a 7-vertex and a 6-vertex, then A sends at most $\frac{5}{21}$ into x_5. If A is adjacent to two 7-vertices, then A sends at most $\frac{3}{21}$ into x_5. In any case, A sends at most $\frac{1}{3}$ into x_5. Clearly, if x_5 is adjacent to at most three 3-faces, $ch'(x_5) \leq 0$. Suppose that the vertices adjacent to x_5 are $y_1, \ldots, y_5$, in a cyclic ordering according the embedding. It is easy to see that each of $y_1, \ldots, y_5$ is an (≥ 5)-vertex, or it would be special. Without loss of generality, for $i \in \{1, 2, 3, 4\}, T_i := x_5, y_i, y_{i+1}$ is a 3-face of G.

Assume that A is adjacent to exactly four 3-faces. It cannot be that both y_1 and y_5 are 5-vertices, or x_5 would be a (5, 7, 7, 7, 5) around a 5 and would be special.

Suppose that $\deg(y_1) = 5$, and $\deg(y_5) = 6$. Since x_5 is neither a (5, 7, 6, 7, 6) around a 5 nor a (5, 7, 7, 6, 6) around a 5, it follows that each of y_2, y_3, y_4 is a 7-vertex. Thus, x_5 receives at most $\frac{2}{7}$ from T_1, at most $\frac{1}{7}$ from each of T_2, T_3, and at most $\frac{5}{21}$ from T_4. Thus, $ch'(x_5) \leq -\frac{4}{21}$.

Suppose that $\deg(y_1) = 5$, and $\deg(y_5) = 7$. Since x_5 is not a (5, 6, 7) around a 5, then $\deg(y_2) = 7$, and either $\deg(y_3) = 7$ or $\deg(y_4) \geq 6$. If $\deg(y_3) = 7$, then x_5 receives at most $\frac{2}{7}$ from each of T_1, T_3, T_4 and at most $\frac{1}{7}$ from T_2, and $ch'(x_5) \leq 0$. If $\deg(y_4) \geq 6$, then x_5 receives at most $\frac{2}{7}$ from T_1, at most $\frac{5}{21}$ from T_2, at most $\frac{1}{3}$ from T_3, and at most $\frac{1}{14}$ from T_4 (since T_4 sends out at least $\frac{1}{6}$ by Rule 2). In this case, $ch'(x_5) \leq -\frac{1}{14}$.

Suppose that $\deg(y_1) \geq 6$, and $\deg(y_5) \geq 6$. Suppose that $\deg(y_1) = 6$. Here, x_5 receives at most $\frac{1}{3}$ from T_2. Note that T_1 sends out at least $\frac{1}{6}$ by Rule 2, and at least $\frac{1}{3}$ by Rule 4. If $\deg(y_2) \geq 6$, then T_1 sends out at least $\frac{2}{3}$ by Rules 3 and 4. If $\deg(y_2) = 5$, then, since y_2 is not a (5, 6, 7) around a 5, the edge y_1y_2 is adjacent to an (≥ 4)-face, and T_1 sends out $\frac{1}{3}$ by Rule 2. In either case, x_5 receives at most $\frac{1}{6}$ from T_1. Also, x_5 receives a total of at most $\frac{1}{2}$ from T_1 and T_2.

Suppose that $\deg(y_1) = 7$. Note that T_1 sends out at least $\frac{1}{6}$ by Rule 2, and at least $\frac{3}{7}$ by Rule 3. If $\deg(y_2) \geq 6$, then T_1 sends out a total of at least $\frac{16}{21}$ by Rules

3 and 4, and x_5 receives at most $\frac{1}{14}$ from T_1. Since x_5 receives at most $\frac{1}{3}$ from T_2, it follows that x_5 receives a total of at most $\frac{17}{42}$ from T_1 and T_2. If $\deg(y_2) = 5$, then, since x_5 is not a (5, 6, 7) around a 5, it follows that $\deg(y_3) = 7$. In this case, x_5 receives at most $\frac{2}{7}$ from T_2. Also, x_5 receives at most $\frac{17}{84}$ from T_1. Thus, x_5 receives a total of at most $\frac{41}{84}$ from T_1 and T_2 in this case.

In any case, x_5 receives a total of at most $\frac{1}{2}$ from T_1 and T_2. Symmetrically, x_5 receives a total of at most $\frac{1}{2}$ from T_3 and T_4. Thus, $ch'(x_5) \leq 0$.

Assume that x_5 is adjacent to five 3-faces. Since x_5 is neither a (6, 6, 7, 7, 7) surrounding a 5 nor a (7, 6, 7, 6, 7) surrounding a 5, x_5 is adjacent to at least four 7-vertices, say $y_1, \ldots, y_4$. Thus, x_5 receives at most $\frac{1}{7}$ from each of T_1, T_2, T_3, and at most $\frac{5}{21}$ from each of the other two 3-faces. In this case, $ch'(x_5) \leq -\frac{2}{21}$.

Let x_6 be a 6-vertex of G. Note that $ch(x_6) = -2$. Since x_6 is adjacent to no 3-vertex, x_6 receives at most $\frac{1}{3}$ from each face adjacent to it, and $ch'(x_6) \leq 0$.

Let x_7 be a 7-vertex of G. Note that $ch(x_7) = -3$. Since no 3-vertex is adjacent to a 3-face, x_7 receives at most $\frac{3}{7}$ from each face adjacent to it, and $ch'(x_7) \leq 0$.

The above argument implies that $\sum_{x \in V(G) \cup F(G)} ch'(x) \leq 0$. On the other hand, since $\sum_{x \in V(G) \cup F(G)} ch(x) = \sum_{x \in V(G) \cup F(G)} ch'(x)$, this contradicts Lemma 3.1. ∎

4. REDUCIBILITY

Let a *minimal* graph be a connected graph on the fewest edges, which has no total 9-coloring. To complete the proof of Theorem 1.1, it suffices to show the following.

Theorem 4.1. *No minimal graph has a special vertex.*

The ideas necessary to prove this are quite straightforward. Let G be a minimal graph with a special vertex x. Refer to the table in Section 2 that lists the thirteen types of special vertices. The second column of the table is what is at times known as a reducer. Say that x is of type k, meaning described by the kth line of the original table. Form a graph H from G by deleting and adding the edges described in the second column on the kth line, labeled according to the definitions of around and surrounding.

Since G is minimal, the graph H has a total 9-coloring χ. Each type of special vertex has certain obvious defining elements associated with it (e.g., the last type has six defining vertices and ten defining edges). The coloring χ induces a *partial total k-coloring* ψ of G, meaning that the defining elements are uncolored, everything else is colored, and two distinct colored elements receive distinct colors. To prove Theorem 4.1, it suffices to show that every such ψ arising from a total 9-coloring of the reducer H extends to a total 9-coloring of G simply by assigning a color to each uncolored (defining) element. This is clearly a finite problem.

As an example, consider the third type, a 4-vertex x adjacent to an (≤ 5)-vertex y. Let $H = G - xy$; let χ be a total 9-coloring of H. This induces a partial total 9-coloring of G with only x, y, and xy uncolored. Color y with $\chi(y)$; this will give a partial total 9-coloring of G with only x and xy uncolored. Note that coloring x with $\chi(x)$ may not give a partial total 9-coloring of G, as x and y are adjacent in G, but not in H. Instead, note that xy has at most eight colored adjacent elements; color it a color different from the colors of these. To finish, color x a color different from the colors of its eight adjacent elements.

Unfortunately, a complete proof of this for all thirteen types is a long, tedious case analysis. A twenty-five page standard proof of this has been completed, but is too lengthy to include in this article. A copy may be found at `http://www.math.gatech.edu/~thomas/DPS/`. As an alternative to checking that, the authors have also written a two page computer program to do the job, available at the same Web site.

The authors hope that the same technique may be used to complete the proof of the total coloring conjecture for planar and projective planar graphs. The reducibility will only increase in difficulty, as this class of graphs includes large graphs almost every vertex of which has degree equal to the maximum degree of the graph. It seems worthy to pursue this, on the other hand, to verify the total coloring conjecture for this important class of graphs.

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